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Notes on dynamics of
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Notes on Dynamics of Rotation

W. J. LOUDON

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Notes on Dynamics of Rotation

by

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Notes on Dynamics of Rotation

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1. Linear Velocity.

Represented in direction and in magnitude by a straight line. If no force acts, then the direction and magnitude remain constant, and $s = vt$.

If either the direction or magnitude changes, or if both change, under the action of force, then $v = \frac{ds}{dt}$.

Illustrations.

2. Statement of the parallelogram of velocities. Illustrations of this principle:

(*a*) Man rowing uniformly across a stream, which flows between parallel banks with uniform or variable velocity.

(*b*) Motion of a projectile, whose initial velocity is 200 feet a second, inclined at an angle of 45° to the horizon.

3. Angular Velocity.

Measured by $\frac{\theta}{t}$ or $\frac{d\theta}{dt}$.

In circular motion $v = r\omega$.

4. Combination of linear and angular velocity.

(Direction and magnitude both change)

$V_1 = \frac{2\pi r}{T} = V$

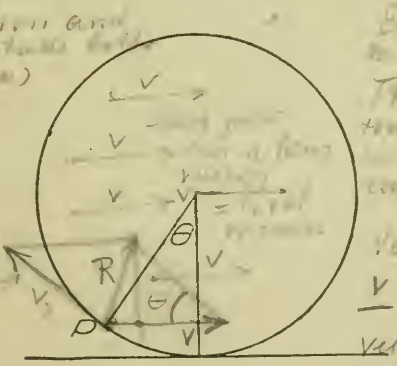
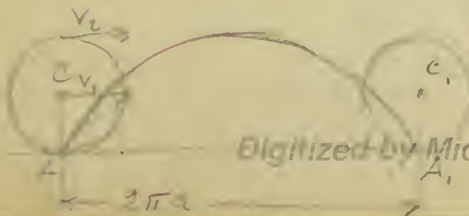


Fig 1

$V = C$

$\frac{V_2}{V_1} = \frac{2r\omega}{r\omega}$

$R = 2\omega r \sin\frac{\theta}{2}$



A wheel of radius a rolls uniformly along a horizontal plane: shew that, at any instant,

(a) the velocity of the centre of the wheel is $a\omega$,
(b) the velocity of the highest point is $2\,a\omega$,
(c) the velocity of the lowest point is zero,
(d) the velocity of the point P in the figure is

$$2\,a\omega \sin\frac{\theta}{2}.$$

5. Examples.

1. A body is projected at an angle of 60° to the horizon with a velocity of 1000 feet a second; find its absolute velocity and position at the end of 5 seconds. Also, find the coördinates of the highest point to which it rises, and the horizontal range.

2. Shew that the path of a projectile is a parabola.

= 4000 m/sec. 3. Find the linear velocity of a point in latitude 43° 39′ due to the earth's rotation on its axis.

4. If the earth be supposed to move in a circle about the sun what is its velocity of translation?

5. A wheel 6 feet in diameter rolls uniformly on a horizontal plane, making 4 revolutions in 5 seconds: find the velocity of a point which is, at any instant, 4 feet above the plane. Also, find the point on the rim of the wheel which has the same velocity (in magnitude) as the centre.

6. A man can throw a ball 50 yards vertically upwards: shew that the greatest horizontal distance he can throw it is 100 yards.

7. A projectile has an initial velocity of 280 ft. per second. Find the angle of elevation in order that it may hit an object whose altitude is 192 ft. and horizontal distance 2240 ft. Also, find the time of flight.

6. Centre of gravity, or centroid.

The point in a body at which the weight may be supposed to act.

Assuming the centre of gravity of a uniform rod to be at the centre, it follows that the centre of gravity of all symmetrical bodies is at the centre of symmetry; where bodies are symmetrical in respect of a line or plane, the centre of gravity must lie in that line or plane.

$v = u + ft$

$\frac{1}{2}(u+v)$

7. *C.G.* of a triangular area: deduce from rod. Found also by experiment.

8. *C.G.* of any body, more particularly if in the form of a flat plate, found by experiment.

9. New conception of *centre of gravity:* the point in a body about which rotation takes place most easily.

Any axis passing through the centre of gravity of a body may therefore be viewed as an *axis of least effort.*

10. General method of finding the centroid of a body or system of bodies.

Experiment with two small spheres joined by a fine wire and suspended by a twisted thread attached to their centroid. This gives an experimental relation upon which the following discussion is based.

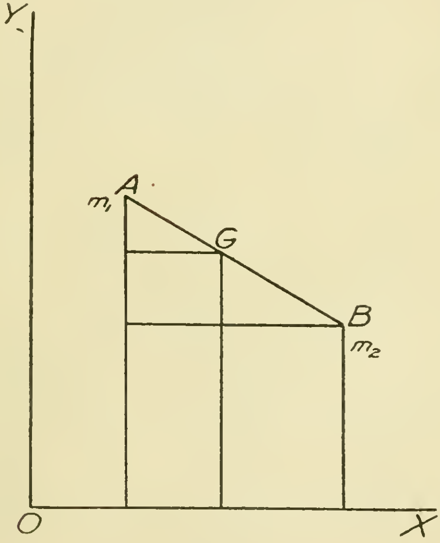


Fig. 2

Let two small masses, m_1, m_2, be at the points A, B: G, their centre of gravity is given by the relation

$$m_1 \times AG = m_2 \times BG.$$

Hence $$m_1(\bar{x}-x_1)=m_2(x_2-\bar{x})$$
the coördinates of A, B, G being, respectively, x_1, y_1; x_2, y_2; $\bar{x}$, $\bar{y}$. Therefore

$$\bar{x}=\frac{m_1x_1+m_2x_2}{m_1+m_2}.$$

Similarly, it may be shewn that

$$\bar{y}=\frac{m_1y_1+m_2y_2}{m_1+m_2}.$$

If there are three small masses m_1, m_2, m_3 at the points x_1, y_1; x_2, y_2; x_3, y_3; then, replacing m_1 and m_2 by a mass m_1+m_2 at G, G', the centre of gravity of the three masses m_1, m_2, m_3 will be defined by the relation

$$(m_1+m_2)\ GG'=m_3\times AG'$$

$$\therefore (m_1+m_2)\left\{\bar{x}-\frac{m_1x_1+m_2x_2}{m_1+m_2}\right\}=m_3(x_3-\bar{x})$$

$$\therefore \bar{x}=\frac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3}.$$

Similarly, $$\bar{y}=\frac{m_1y_1+m_2y_2+m_3y_3}{m_1+m_2+m_3}.$$

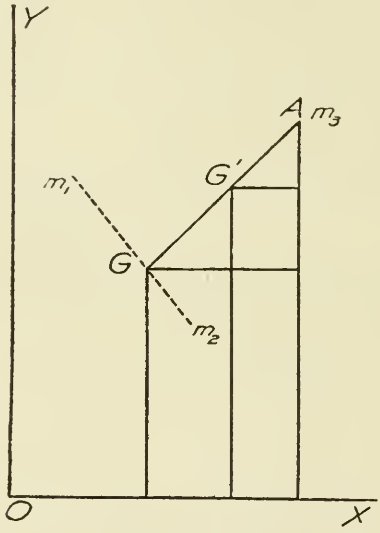


Fig. 3

Therefore, generally, it may be shewn that for any number of masses $m_1, \ldots . m_n$, the centre of gravity is defined by the relations

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \ldots . etc.}{m_1 + m_2 + \ldots . etc.} = \frac{\Sigma(mx)}{\Sigma(m)},$$

$$\bar{y} = \frac{m_1y_1 + m_2y_2 + \ldots . + etc.}{m_1 + m_2 + \ldots . + etc.} = \frac{\Sigma(my)}{\Sigma(m)}.$$

These expressions are true where $m_1, m_2, \ldots .$ are infinitely small, or where they represent finite masses collected at individual centres of gravity. They may also be used to determine the centroid of a body from which one or more portions are removed. In this case some of the masses are negative.

11.

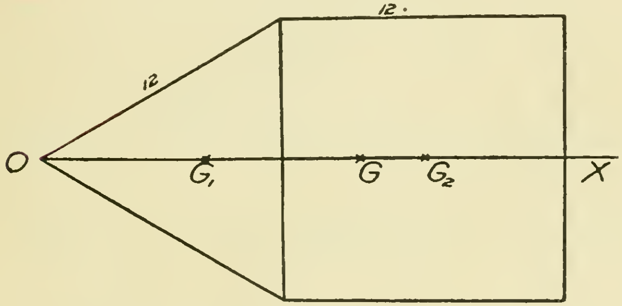


Fig. 4

Examples.

1. To find the centre of gravity of a plate in the form of the figure shewn: a square joined to an equilateral triangle.

The centre of gravity must lie on OX, the line of symmetry. Collect the mass of the triangle at G_1 and the mass of the square at G_2. Then, if G be the centroid required,

$$\bar{x} = OG = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} =$$

$$\frac{36\sqrt{3} \times 4\sqrt{3} + 144 \times 6\,(1 + \sqrt{3})}{36\sqrt{3} + 144} = 13.5.$$

2.

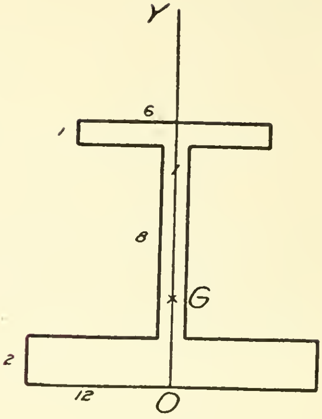


Fig 5

The centre of gravity of the section shewn here is on the line of symmetry *OY*.

$$\bar{y} = OG = \frac{12\times2\times1+8\times1\times6+6\times1\times10.5}{12\times2+8\times1+6\times1} = 3''.55.$$

3.

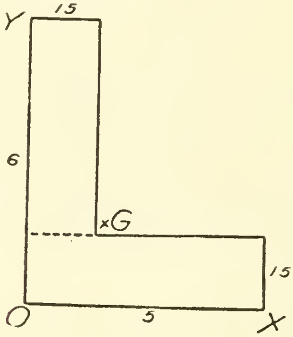


Fig 6

The centre of gravity of this section will be given by

$$\overline{x} = \frac{5\times1.5\times2.5+4.5\times1.5\times.75}{5\times1.5+4.5\times1.5} = 1.67.$$

$$\overline{y} = \frac{5\times1.5\times.75+4.5\times1.5\times3.75}{5\times1.5+4.5\times1.5} = 2.17.$$

The position of the point $\overline{x}$, $\overline{y}$ is shewn at G.

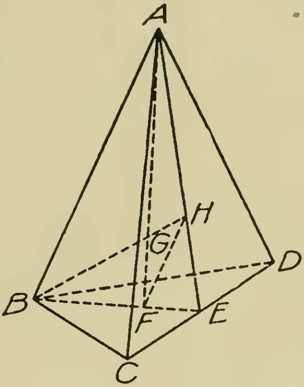


Fig. 7

4. The centre of gravity of a tetrahedron $ABCD$ is determined by dividing it up into triangular plates parallel to BCD: the centre of gravity thus lying on AF, where F is the centre of gravity of BCD. Similarly the centre of gravity of the tetrahedron lies on BH where H is the centre of gravity of ACD: G is the required $C.\ G.$

$$\text{and } \frac{FH}{AB} = \frac{FG}{GA} = \frac{EF}{EB} = \tfrac{1}{3}$$

$$\therefore\ FG = \tfrac{1}{3}GA$$

$$\therefore\ FG = \tfrac{1}{4}FA$$

N.B.—Since any pyramid, or, in the limiting case, a cone, may be divided up into a large number of tetrahedra, it follows that the centre of gravity of any such pyramid or cone lies at a distance from the base one fourth of the whole distance from the apex to the base.

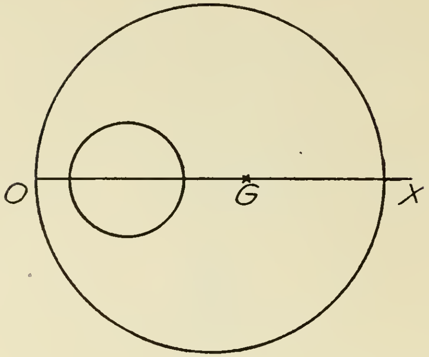


Fig 8

5. A circular area of radius b is cut from a circular area of radius a, to find the centroid of the portion left.

This problem illustrates the use of a negative mass, the portion removed being considered as negative in the formula.

If the radius of the larger circle be a, of the smaller one b, and if the distance between their centres be c, then the centroid required is at the point G, to the right of the centre of the larger circle, on the line of symmetry OX, such that

$$\bar{x} = OG = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{\pi a^2 . a - \pi b^2 . (a-c)}{\pi(a^2 - b^2)}$$

$$= \frac{a^3 - ab^2 + b^2c}{a^2 - b^2}$$

12. *Geometrical methods* of finding the centres of gravity of some figures bounded by curves.

(*a*) *A semi-circular arc* (radius a).

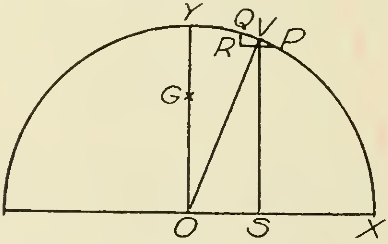


Fig 9

G will evidently lie on the line of symmetry, OY. If PQ be a small element of the arc

$$\bar{y}=OG=\frac{\Sigma\,(my)}{\Sigma\,(m)}=\frac{\Sigma\,(PQ\times y)}{\pi a}$$

considering the density of the arc as unity.

$$\text{But }\cos RPQ=\frac{PR}{PQ}$$

$$\cos SVO=\frac{SV}{VO}$$

V being the centre of PQ.
And the angles RPQ, SVO are equal.

$$\therefore\ \frac{PR}{PQ}=\frac{SV}{VO}=\frac{y}{a}$$

$$\therefore\ PQ=\frac{a}{y}\times PR$$

$$\therefore\ \bar{y}=\frac{\Sigma\,(y\times\frac{a}{y}\times PR)}{\pi a}=\frac{a\Sigma\,(PR)}{\pi a}=\frac{2a^2}{\pi a}=\frac{2a}{\pi}$$

(b) Semi-circular plate (radius a).

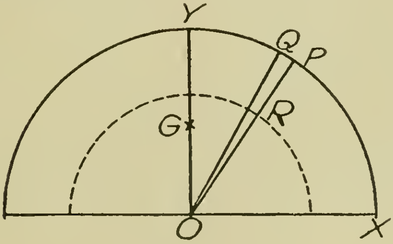


Fig. 10

Dividing the circular plate up into an indefinite number of infinitely small triangular plates, such as OPQ, it is evident that these triangles may be replaced by their masses (all equal), collected at their centres of gravity.

The semi-circular plate may thus be replaced by a series of masses lying on a semi-circle of radius OR, equal to $\frac{2}{3}a$.

But by (a) the centre of gravity of this arrangement is given by

$$\bar{y}=OG=\frac{2\,(\frac{2}{3}a)}{\pi}=\frac{4a}{3\pi}$$

(*c*) A hollow hemispherical shell (radius a).

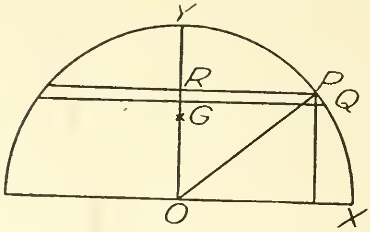


Fig. 11

Divide the shell up into an infinite number of infinitely small annuli by planes parallel to the base, so that the distances cut off by OY are each equal to h.

Then the surface of each annulus shewn in the figure is equal to

$$2\pi \times PR \times PQ$$

$$\text{But } \frac{h}{PQ} = \frac{PR}{OP} \qquad \therefore PQ = h \times \frac{OP}{PR}$$

$$\therefore \text{surface} = 2\pi \times PR \times h \times \frac{OP}{PR} = 2\pi a h.$$

And this is constant.

Therefore the surface of the shell may be replaced by a number of equal small masses each collected at its centre of gravity, which lies on OY.

The centre of gravity of the shell is thus the same as that of a number of equal masses distributed uniformly along Oy.

$$\therefore OG = \bar{y} = \frac{a}{2}$$

(*d*) Solid hemisphere (radius a).

The centre of gravity will lie on the axis of symmetry. Dividing the hemisphere up into an infinite number of infinitely small equal cones with a common apex at the centre of the hemisphere, it is evident that, since each cone has its centre of gravity at a distance $\frac{3}{4}a$ from the apex, the hemisphere may be replaced by a series of particles arranged on a hemispherical shell of radius $\frac{3}{4}a$. Therefore the centre of gravity of the hemisphere is $\frac{1}{2} \times \frac{3}{4}a$ or $\frac{3}{8}a$ from the centre.

(*e*) Hollow conical shell.

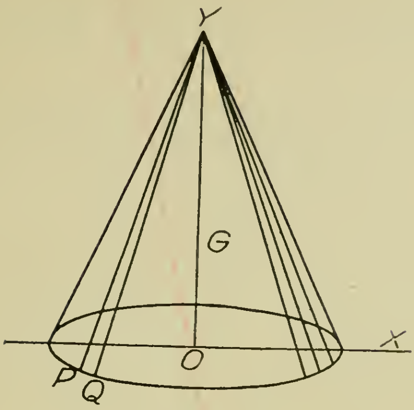


Fig. 12

If the surface of the shell be divided into an indefinite number of infinitely small equal triangular areas, such as YPQ, the centre of gravity of each of which is $\frac{2}{3}$ the distance from Y to the rim of the cone, it will be seen that the centre of gravity of the whole surface is the same as that of a circle of equal masses, arranged symmetrically about OY. The centre of gravity of the shell is therefore at the centre of this circle, that is at G, such that

$$\bar{y} = OG = \tfrac{1}{3}h$$

if h be the height of the cone.

13. Determination of the centroid of a body by the Integral Calculus. (See Appendix, p. 39).

As an illustration of the method to be employed, take the quadrant of a circular area, radius a.

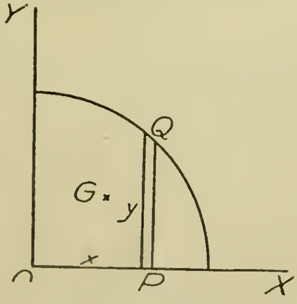


Fig 13

Divide the area up into infinitely small areas, such as PQ, each equal to $y\Delta x$: x, y, being the coördinates of any point Q.

$$\text{Then } y = \sqrt{a^2 - x^2}$$

$$\bar{x} = \frac{\Sigma\,(mx)}{\Sigma\,(m)} = \frac{\Sigma\,(y\Delta x \times x)}{\frac{1}{4}\pi a^2}$$

Taking the density as unity.

$$\therefore \bar{x} = \frac{\int_o^a xydx}{\frac{1}{4}\pi a^2} = \frac{\int_o^a x\sqrt{a^2 - x^2}dx}{\frac{1}{4}\pi a^2} = \frac{4a}{3\pi}$$

Similarly, by dividing up the area into elements by lines drawn parallel to OX, or by symmetry, it may be seen that

$$\bar{y} = \frac{4a}{3\pi}$$

It may be inferred from these results that the centroid of a semi-circular area is on the line of symmetry at a distance $\frac{4a}{3\pi}$ from the centre.

14. The term centroid is also applied to geometrical areas or sections where there is no surface density. In this case the position of the centroid is determined by the relation:

$$\bar{x} = \frac{\Sigma x dA}{A}$$

where dA is an infinitesimal portion of the area or section A.

(See Sec. 11, Examples 2, 3.)

15. *Stability* dependent on the position of the centre of gravity.

Illustrations of stable, unstable, and neutral equilibrium.

Conditions of Stability.

16. When a body moves from a higher position to a lower the work done by gravity is equal to the weight of the body multiplied by the *vertical* displacement of its centroid. If it moves from a lower position to a higher

then the work done against gravity is measured in the same way.

Proof.

If a single particle m moves from a higher position A to a lower position B, along any path whatever, the total work done on m by gravitation is equal to

$$\Sigma mg \cos \theta \, . \, \Delta s$$

where θ is the angle between any tangent to the path and the vertical, and Δs is an element of the path. This quantity is equal to mgy, where y is the vertical displacement of m measured from a horizontal axis. What is true in the case of one particle will be true for any system of particles. Therefore the total work on any body or system of bodies is equal to $\Sigma(mgy)$, which, by definition of the centroid, is equal to Mgy: that is, the weight of the body multiplied by the vertical displacement of its centroid.

17. **Examples.**

1. A cross is formed of six equal squares joined together: find the position of the centre of gravity.

2. A circular lamina, 4 inches in diameter, has two holes cut out of it, one 1.5 inches in diameter and the other 1 inch in diameter, with their centres 1 inch and 1.25 inches, respectively, from the centre of the lamina, and situated on diameters mutually perpendicular. Find the centre of gravity of the portion left.

Ans. 0 202 inches from the centre.

3. If two perpendicular tangents be drawn to a circle 4 feet in diameter, find the distance of the centre of gravity of the area inclosed between the tangents and the arc from either tangent.

Ans. 5.36 inches.

4. A solid of uniform material consists of a cylinder 4 inches in diameter and 10 inches in length, with a hemispherical end, the circular face of which coincides with one end of the cylinder. The other end of the cylinder is pierced by a cylindrical hole, 2 inches in diameter, extending to a depth of 7 inches along the cylinder and co-axial with it. Find the centre of gravity of the solid.

Ans. 6.08 inches from the flat end.

5. A solid consists of a hemisphere and a cylinder, each 10 inches in diameter, the centre of the base of the

hemisphere being at one end of the axis of the cylinder. What is the greatest length of cylinder consistent with stability, when the solid rests with its curved end on a horizontal plane?

Ans. $\frac{5}{\sqrt{2}}$ inches.

6. A uniform chain, 40 feet long, and weighing 10 lbs. per foot, hangs vertically. How much work is necessary to wind it up?

Ans. 8000 foot pounds.

7. A cubical block of stone, whose edge is 3 feet, rests with one face on the ground: the material weighing 150 lbs. per cubic foot. How much work is necessary to tilt the block into a position of unstable equilibrium resting on one edge?

Ans. 2514 foot pounds.

8. If $ABCD$ be a tetrahedron, and if a plane CDE, passing through the edge CD, cuts AB in E, prove that the line joining the centres of gravity of the tetrahedra $ABCD$, $AECD$ is parallel to AB.

9. If a right-angled triangular lamina be suspended from one of the points of trisection of the hypothenuse, shew that it will rest with one side horizontal.

10. A square plate is divided into two equal parts by a line drawn through the centre, making an angle θ with the sides. Shew that, if the origin be chosen at the corner furthest from the line, the coördinates of the centroid of either half will be

$$x = \frac{a}{12}(3+\cot^2\theta)$$

$$y = \frac{a}{6}(3-\cot\theta).$$

11. A closed vessel is formed by a thin right circular cylinder, one end of which is closed by a hemispherical cap and the other by a flat plate, the whole being of uniform thickness. Shew that the vessel will rest in neutral equilibrium anywhere on its curved surface if the height of the cylinder is to the radius of the end in the ratio of $\sqrt{5}-1$ to 2.

12. A solid cone and hemisphere are joined together, their bases in contact and of equal area. Shew that the body thus formed will rest in equilibrium in any position

if the height of the cone is to the radius of the hemisphere in the ratio of $\sqrt{3}$ to 1.

13. A metal plate composed of an isosceles triangle (vertical angle 2α) and a semi-circular area on the same base is placed in a vertical plane with its curved rim resting on a horizontal plane: shew that it will rest in any position if $\tan \alpha = \dfrac{1}{\sqrt{2}}$.

14. Find, by the method of the calculus, the centroids of:

1. Any circular arc; $\left(\dfrac{r \sin \alpha}{\alpha}\right)$
2. A sector of a circle; $\left(\dfrac{2}{3} \cdot \dfrac{r \sin \alpha}{\alpha}\right)$
3. A parabolic area; $\left(\dfrac{3}{5}\, h\right)$
4. A paraboloid of revolution; $\left(\dfrac{2}{3}\, h\right)$
5. A hemispherical shell; $\left(\dfrac{r}{2}\right)$
6. A solid cone; $\left(\dfrac{h}{4}\right)$
7. A solid hemisphere; $\left(\dfrac{3}{8}\, r\right)$
8. A quadrant of an elliptic area; $\left(\dfrac{4a}{3\pi}, \dfrac{4b}{3\pi}\right)$
9. The area included by the curves $y^2 = ax$ and $x^2 = by$.

$$\bar{x} = \frac{9}{20} \sqrt[3]{ab^2}, \quad \bar{y} = \frac{9}{20} \sqrt[3]{a^2 b}.$$

10. The area contained between the parabola $y^2 = ax$ and the circle $y^2 = 2ax - x^2$, which is above the axis of x.

$$\bar{x} = \frac{11}{25} a, \quad \bar{y} = \frac{7}{10} a \text{ nearly.}$$

18. If any area A revolves about an external axis in its own plane the volume of the solid generated is $2\pi A\bar{x}$ where $\bar{x}$ is the distance of the centroid of A from the axis. Also, if any curve of length L revolves about an external axis in the same plane the surface generated is equal to $2\pi L\bar{x}$ where $\bar{x}$ is the distance of the centroid of L from the axis.

Let dA be an element of the area A. Then the volume generated by its revolution will be $2\pi x dA$, x being its distance from the axis.

Therefore the total volume will be $\Sigma(2\pi x dA) = 2\pi\Sigma x dA$.

But, by definition of C. G. of A,

$$A\bar{x} = \Sigma x dA$$

$$\therefore \text{ total volume} = 2\pi A\bar{x}.$$

Also, let ds be an element of L. Then the surface generated by it will be $2\pi x ds$, x being its distance from the axis.

Therefore the total surface will be $\Sigma(2\pi x ds) = 2\pi\Sigma x ds$.

But, by definition of the C. G. of L,

$$L\bar{x} = \Sigma x ds$$

$$\therefore \text{ total surface} = 2\pi L\bar{x}$$

These formulas are true when the axis forms a boundary of the curve or area and when the lines of the boundary are straight or broken.

19. Examples.

1. A right angled triangle revolves about one side: find the volume and surface of the cone generated.

2. Find the volume and surface of an anchor ring.

3. Given the volume of a sphere, deduce the centroid of a semi-circular area.

4. Given the surface of a sphere, deduce the centroid of a semi-circular arc.

5. A segment of a circle revolves around its chord which subtends an angle of 90° at the centre: find the surface and volume of the solid generated

$$1.9a^2, \quad .21a^3.$$

6. If the area in Sec. 17, Ex. 14(10), revolves about the axis of y completely, shew that the centroid of the solid thus formed will be given by

$$\bar{y} = \frac{35}{44}\, a \text{ nearly.}$$

20. Kinetic Energy of a Rotary Body.

When a body rotates about a fixed axis each small element m, at a distance r from the axis of rotation, has a linear velocity $r\omega$, ω being the angular velocity at any instant. Therefore its energy is $\frac{1}{2}mr^2\omega^2$; and the total energy of the body will be

$$\Sigma(\tfrac{1}{2}mr^2\omega^2) = \tfrac{1}{2}\omega^2\Sigma mr^2$$

since ω is the same for all elements.

Thus at any given time the energy is equal to

$$\tfrac{1}{2}I\omega^2$$

where I represents the quantity Σmr^2, and is termed a *moment of inertia.*

If the body be rotating uniformly then the energy remains constant. If not, then ω is continually changing but the expression for the energy at any given instant still assumes the form given, ω being the angular velocity at the time in question.

21. Moments of Inertia.

It will thus be seen that the energy of a rotating body may be calculated if we know its angular velocity and its moment of inertia.

The determination of the former is simple and involves only arithmetical computation; but the determination of a moment of inertia involves the summation of a series of infinitely small terms: and this involves the methods of the integral calculus.

The following illustrations will serve to shew how these moments of inertia may be most easily determined.

22. A circular plate, radius a, revolves about an axis through its centre of gravity perpendicular to the plate, with uniform angular velocity (ω): to calculate its moment of inertia and kinetic energy.

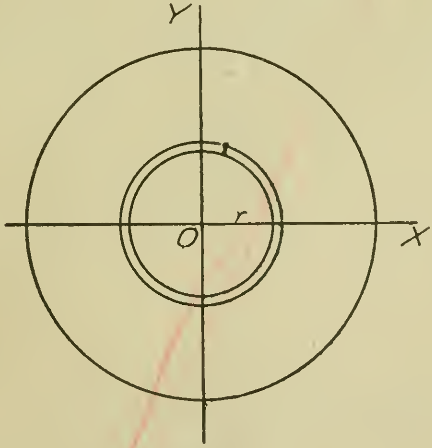


Fig. 14

Divide the circular area up into annuli, each of infinitely small width Δr, of which one is shewn in the figure, of radius r.

Each portion of this annulus is at the same distance r from the axis of rotation, which is perpendicular to the plate through O.

The moment of inertia of this annulus is therefore equal to $m \times 2\pi r \Delta r \times r^2$, the density being taken as m. Therefore the total moment of inertia of the whole plate is

$$\Sigma(2\pi r \Delta r \times r^2 \times m) = m \,.\, \Sigma(2\pi r^3 \Delta r)$$

$$= m \int_0^a 2\,\pi r^3 dr = 2\,\pi m \int_0^a r^3 dr = 2\pi m \,.\, \frac{a^4}{4}$$

$$= M \,.\, \frac{a^2}{2}$$

if M be the mass of the plate.

The energy of rotation is equal to

$$\tfrac{1}{2}\Sigma(mr^2\omega^2) = \tfrac{1}{2}\omega^2 \,.\, I$$

$$\text{But } I = M \,.\, \frac{a^2}{2}$$

$$\therefore \text{energy} = \tfrac{1}{2} \,.\, \omega^2 \,.\, M \,.\, \frac{a^2}{2}$$

$$= \omega^2 M \, \frac{a^2}{4} = M \,.\, \frac{a^2\omega^2}{4}\,.$$

23. A straight uniform rod rotates about an axis through its centre, perpendicular to itself, with uniform angular velocity, ω: to determine the moment of inertia and the kinetic energy.

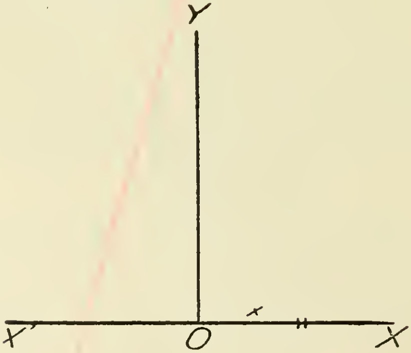


Fig 15

Let the rod, XX', be divided up into an infinite number of infinitely small elements, each equal to $m\Delta x$, the density being m.

Then the moment of inertia of the whole rod about OY is equal to

$$\Sigma(m\Delta x \times x^2)$$

This being calculated for both sides of the origin.

Hence

$$I = 2\int_o^a mx^2 dx$$

If the length of the rod be $2a$.

$$\therefore I = 2m \, . \, \frac{a^3}{3} = M \, . \, \frac{a^2}{3}$$

If M be the mass of the rod.

Therefore the kinetic energy is equal to

$$\tfrac{1}{2}\omega^2 \, . \, \Sigma mr^2 = \tfrac{1}{2}\omega^2 \, . \, M\frac{a^2}{3} = \omega^2 \, . \, M \frac{a^2}{6}$$

24. Although moments of inertia may be always found by the process of integration, yet much labour may be saved by the following propositions.

Proposition I.

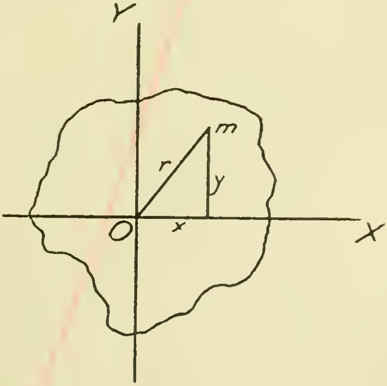


Fig. 16

Let there be any *thin plate*, of any shape, as shown in the figure, and let O be any point in its plane; OX, OY, two rectangular axes.

Then the moment of inertia of the plate about an axis through O perpendicular to the plate is equal to $\Sigma(mr^2)$, where m is any small mass of the plate situated at the point x, y.

But since $r^2 = y^2 + y^2$, we have

$$\Sigma mr^2 = \Sigma mx^2 + \Sigma my^2.$$

That is, the moment of inertia about the line through O perpendicular to the plate is equal to the sum of the moments about the two rectangular axes. This proposition may be expressed in the form

$$I = I_y + I_x .$$

Proposition II.

If, for any body, I be the moment of inertia about any axis, and I_G be the moment of inertia about a parallel axis through the centre of gravity, and p be the perpendicular distance between the two axes, then

$$I = I_G + Mp^2$$

where M is the mass of the body.

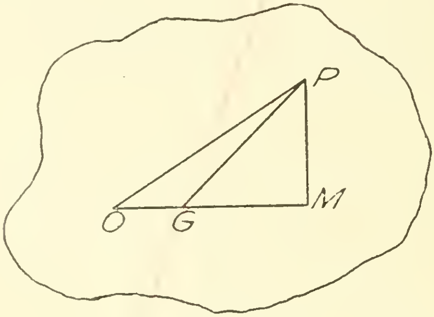


Fig. 17

Take any section through the body perpendicular to the two axes, and let this section contain the centre of gravity of the body. Then, if P be any point in this section, join OP, GP, and complete the figure: G being the centre of gravity and $OG = p$.

Then the moment of inertia about the axis through O, perpendicular to the section is equal to $\Sigma(m \times OP^2)$.

But $OP^2 = OG^2 + GP^2 + 2\,OG\,.\,GM.$

$$\therefore \Sigma(mOP^2) = \Sigma(mOG^2) + \Sigma m(GP^2) + 2OG\,.\,\Sigma(m\,.\,GM)$$

$$\therefore I = I_G + Mp^2$$

Since $\Sigma(m \times GM) = O$, by definition of the centre of gravity.

N.B.—What is true of the section through the centre of gravity is true also of all parallel sections, and therefore the above formula is true of the entire body.

25. *Illustrative Examples.*

(1) Find the moment of inertia of a circular plate about any diameter.

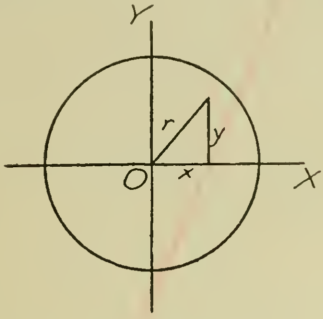


Fig. 18

Since the moment of inertia of the plate about an axis through O perpendicular to the plate is $M\frac{a^2}{2}$, we have

$$M\frac{a^2}{2} = \Sigma mr^2 = \Sigma mx^2 + \Sigma my^2$$

when r is the distance from O to any point x, y.

But $\Sigma mx^2 = \Sigma my^2$, by symmetry;

$$\therefore \Sigma mx^2 = \Sigma my^2 = \tfrac{1}{2} \,.\, M\frac{a^2}{2} = M\frac{a^2}{4}$$

(2) Find the moment of inertia of a circular plate about a tangent at any point on the edge.

If I be the required moment of inertia and I_G the moment of inertia about a parallel line through the centre of the plate,

$$\therefore I = I_G + Ma^2$$

where a is the radius of the circle.

$$\text{And } I_G = M\frac{a^2}{4}$$

$$\therefore I = M\frac{a^2}{4} + Ma^2 = M \,.\, \frac{5a^2}{4}$$

(3) The moment of inertia of a circular plate about a line through any point of the edge, perpendicular to the plate is

$$M\frac{a^2}{2} + Ma^2 = M\frac{3a^2}{2}$$

(4) The moment of inertia of a rectangular plate, sides $2a$, $2b$, about a line through the centre, perpendicular to the sides $(2a)$, is equal to $M\frac{a^2}{3}$; about a line through the centre, perpendicular to the sides $(2b)$, is equal to $M\frac{b^2}{3}$; and about a line through the centre, perpendicular to the plate itself, is $M\frac{a^2+b^2}{3}$

Also, the moments of inertia of the plate about its two edges are respectively

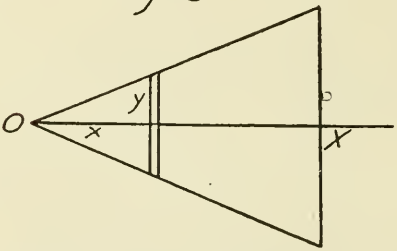
$$M\frac{4a^2}{3},\ M\frac{4b^2}{3}$$

For a square plate, $a = b$.

The above results are obtained by dividing the plate into an infinite number of infinitely small rods.

(5) To find the moment of inertia of a solid right circular cone about its axis.

Fig 19



Let the height of the cone be h, the radius of the base b, and the density m. And let the cone, of which the section is shewn in the figure, be divided into an infinite number of slices each of width Δx.

The moment of inertia of the cone about its axis, OX, is equal to

$$\int_0^h \pi y^2 . m . dx . \frac{y^2}{2} = \frac{\pi m}{2}\int_0^h y^4 dx$$

$$\text{But } \frac{y}{x} = \frac{b}{h} \text{ and } \therefore\ y = \frac{b}{h}x$$

$$\therefore I = \frac{\pi m}{2} \cdot \frac{b^4}{h^4}\int_0^h x^4 dx$$

$$= M \cdot \tfrac{3}{10}b^2.$$

(6) The moment of inertia of a sphere, of radius a, about any diameter.

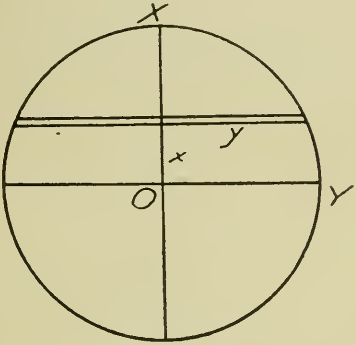


Fig. 20

Let the sphere be divided into an infinite number of slices, perpendicular to the diameter OX, each of thickness Δx.

Then

$$I \text{ (about } OX) = 2\int_0^a \pi y^2 m dx \cdot \frac{y^2}{2}$$

m being the density.

$$\therefore I = \pi m \int_0^a y^4 dx = \pi m \int_0^a (a^2 - x^2)^2 dx$$

Since $y^2 + x^2 = a^2$

$$\therefore I = M \cdot \tfrac{2}{5}a^2.$$

(7) The moment of inertia of an elliptic plate, whose axes are $2a$, $2b$, about the major axis is $M\frac{b^2}{4}$; about the minor axis $M\frac{a^2}{4}$; and, about an axis through the centre, perpendicular to the plate, $M\frac{a^2+b^2}{4}$.

(8) The moment of inertia of a hemisphere about any diameter of its base is evidently half what it would be for the complete sphere and is therefore $M \cdot \frac{2}{5}a^2$ where M is the mass of the hemisphere.

(9) A spherical shell of radii R, r about any diameter.

$$I = M \cdot \frac{2}{5}\,\frac{R^5 - r^5}{R^3 - r^3}.$$

(10) The moment of inertia of a circular cylinder, of length l and radius b, about its axis is $M\frac{b^2}{2}$; and about a line through its centre of gravity perpendicular to the axis is $\frac{M}{4}\left(\frac{l^2}{3} + b^2\right)$.

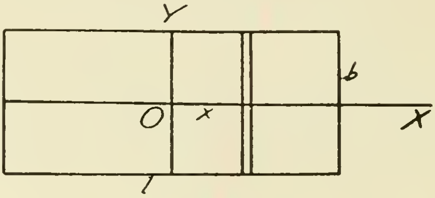


Fig. 21

The first result follows from the moment of inertia of a circular plate about an axis through its centre of gravity perpendicular to the plate.

The second result is obtained as follows:

Let the cylinder be divided into slices, parallel to the given line, and perpendicular to the axis of the cylinder.

Then I (about OY)

$$= 2\int_0^{\frac{l}{2}} \pi b^2 m\,dx \left(\frac{b^2}{4} + x^2\right)$$

$$= 2\pi b^2 m \int_0^{\frac{l}{2}} \left(\frac{b^2}{4} + x^2\right) dx$$

$$= M\left(\frac{b^2}{4} + \frac{l^2}{12}\right)$$

m being the density of the cylinder.

(11) A plane figure consists of a rectangle 8 inches by 4 inches, with a rectangular hole 6 inches by 2 inches, cut so that the diagonals of the two rectangles are in the same straight lines.

Find the moment of inertia of the figure and the radius of gyration about one of the short outer sides.

N.B.—If $\Sigma mr^2 = Mk^2$, then k is termed a *radius of gyration.*

In this particular case $k = 4.76$ inches.

(12) Find the moment of inertia of the area inclosed between two concentric circles of 10 inches and 8 inches diameter respectively, about a diameter.

(13)

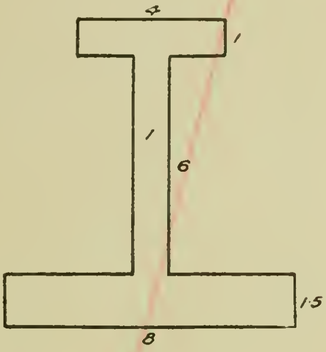


Fig. 22

Shew that the radius of gyration of the section (Fig. 22) about the lowest horizontal edge is 4.29 inches.

(14) The moment of inertia of a grindstone 3 feet in diameter and 8 inches thick about its axis of rotation is 709.3 (lb–ft²) units, the density of the material being 2.14.

(15) The kinetic energy of the stone in the preceding example, rotating with an angular velocity of 5 revolutions in 6 seconds, is 303.7 foot pounds.

To solve this we have:

$$E = \tfrac{1}{2} I\omega^2$$
$$= \tfrac{1}{2} \times 709.3 \times \left(\frac{5 \times 2\pi}{6}\right)^2 \div 32.$$

(16) A door 7.5 feet high and 3 feet wide, weighing 80 lbs., swings on its hinges so that the outward edge moves at the rate of 8 feet per second. How much work must be expended in stopping it?

$$E=\tfrac{1}{2}I\omega^2=\tfrac{1}{2}\times\frac{80}{32}\times\frac{2.25\times4}{3}\times\left(\frac{8}{3}\right)^2=26.7 \text{ ft. pds.}$$

(17) To find the moment of inertia of a right angled triangular area about either side.

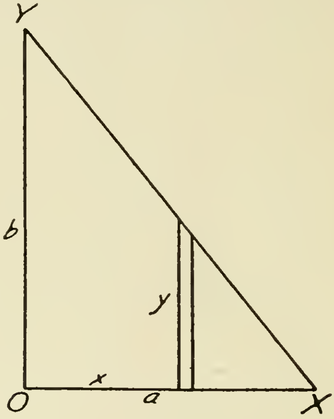


Fig. 23

Divide the area into strips of width Δx. Then, if the sides be a, b, and the density be m,

$$I \text{ about } OY=\int_o^a mydx\,.x^2.$$

But, from the figure, $y=\frac{b}{a}(a-x)$

$$\therefore I=\frac{mb}{a}\int_o^a(ax^2-x^3)dx=M\,.\frac{a^2}{6}$$

if M be the mass of the triangular area.

Similarly it may be shewn that the moment of inertia about OX is $M \cdot \frac{b^2}{6}$.

Also, the moment of inertia about an axis through O, perpendicular to the plate, is $M\left(\frac{a^2+b^2}{6}\right)$.

(18) The moment of inertia of a square plate about a diagonal is $M\frac{a^2}{3}$, if the side of the plate be $2a$.

26. It is evident from the formula $I = I_G + Mp^2$ that the moment of inertia about any line through the centroid of a body is less than that about any parallel line. Consequently the energy required to give a body an angular velocity about an axis through the centroid must be less than that required to rotate it about any parallel axis. This corresponds with the definition of centroid given in Sec. 9.

Of all lines drawn through the centroid there are *three* which are termed *principal axes.*

These are directions such that two of them represent the greatest and least moments of inertia while the third is a mean between the other two.

Principal axes at the centroid are lines of symmetry and may usually be found by inspection or by experiment.

27. When a body moves about an axis with *varying* angular velocity the value of the angular acceleration and the time of describing any angle may be found from the relation between work and energy.

The centroid may or may not lie on the axis of rotation: thus producing two distinct cases. The examples given in the following sections shew how problems involving a *change* of angular velocity may be solved.

28. The Simple Pendulum.

This consists of a fine string, whose mass is supposed to be negligible. One end is fixed, and to the other end is attached a small heavy mass which is allowed to move to and fro under the action of gravity.

Here the centroid of the small mass lies outside of the axis of rotation, and gravity therefore produces a moment or turning effect about the axis.

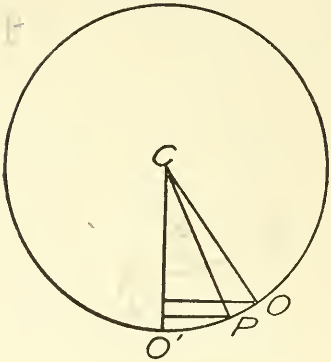


Fig. 24

Let CO be a simple pendulum, of length l, and let it start from O, where the angle $OCO' = \alpha$.

When it reaches the point P, under the action of gravity, where $PCO' = \theta$, we have, by the equivalence of work and energy, the following relation:

$$\tfrac{1}{2}mv^2 = mg\{(l - l\cos\alpha) - (l - l\cos\theta)\}$$

$$\therefore v^2 = 2gl\{\overline{1-\cos\alpha} - \overline{1-\cos\theta}\} = 2gl(\cos\theta - \cos\alpha)$$

$$\therefore v = \sqrt{2gl}\,.\sqrt{\cos\theta - \cos\alpha}$$

which determines the linear velocity at any point, when the corresponding angles θ, α, are given.

The velocity at the lowest point O' is given by

$$v = \sqrt{2gl}\sqrt{1-\cos\alpha} = 2\sin\frac{\alpha}{2}\sqrt{gl}.$$

If α, θ both be very small, the relation becomes

$$v^2 = 2gl\left(2\sin^2\frac{\alpha}{2} - 2\sin^2\frac{\theta}{2}\right)$$

$$\therefore v^2 = gl(\alpha^2 - \theta^2)$$

$$v = -\sqrt{gl}\,\sqrt{\alpha^2 - \theta^2}$$

the negative sign being taken to indicate that the motion is towards O'.

This gives the actual velocity at the point P, and hence, if Δt be a small increment of time after the pendulum passes the point P, and $l\Lambda\theta$ be the corresponding small distance traversed, we must have

$$l\Lambda\theta = -\sqrt{gl}\,\sqrt{\alpha^2-\theta^2}\,.\,\Lambda t$$

and therefore

$$\Lambda t = -\frac{l\Lambda\theta}{\sqrt{gl}\,\sqrt{\alpha^2-\theta^2}}$$

$$= -\sqrt{\frac{l}{g}}\,.\,\frac{\Lambda\theta}{\sqrt{\alpha^2-\theta^2}}$$

Consequently the total time from O to O', when the angles α, θ are small, is

$$-\int_\alpha^o \sqrt{\frac{l}{g}}\,\frac{d\theta}{\sqrt{\alpha^2-\theta^2}} = \sqrt{\frac{l}{g}}\int_o^\alpha \frac{d\theta}{\sqrt{\alpha^2-\theta^2}}$$

$$= \frac{\pi}{2}\sqrt{\frac{l}{g}}.$$

And, since the pendulum after reaching O' evidently has sufficient energy to carry it to O'', such that $O'\,O'' = O\,O'$, and will therefore *oscillate* between O and O'', we must have the total time of a complete oscillation given by

$$T = 4\times\frac{\pi}{2}\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{l}{g}}.$$

29. Experimental proof of the truth of this formula. Practical applications. Determination of g. Variation of value of g at different points of earth's surface.

30. *Illustrative examples.*

(1) Find the length of a seconds pendulum at Toronto, where $g = 32.08$.

(2) A seconds pendulum is found to lose 48.6 seconds in a day at the summit of a mountain; find the height of the mountain.

Ans. 2.25 miles.

(3) Find the length of a pendulum which beats half-seconds at Greenwich ($g = 32.19$).

Ans. 9.78 inches.

(4) The length of the second pendulum at the equator is 39.014 ins.; find the local acceleration of gravity.

Ans. $g = 32.075$.

(5) A seconds clock gains 15 seconds in a day: shew that, in order to beat accurately, the pendulum must be lengthened .0135 in.

31. *The Compound Pendulum.*

Any body capable of oscillating about a fixed horizontal axis, under the action of gravity, may be termed a compound pendulum.

Referring to Fig. 24, it may be seen that if we represent the body by G, its centre of gravity, at a distance h from the fixed axis, the equation of energy becomes

$$\tfrac{1}{2}\Sigma mv^2 = Mgh\left\{cos\ \theta - cos\ a\right\}$$

M being the total mass of the body, and h replacing l in the circle shewn.

$$\text{But } v = r\omega$$

$$\therefore\ \tfrac{1}{2}\Sigma mr^2\omega^2 = Mgh\left\{cos\ \theta - cos\ a\right\}$$

When θ, a, are both small, this becomes

$$\omega^2 = \frac{gh}{h^2+k^2}\,(a^2-\theta^2)$$

which gives the angular velocity at any time. Following a similar line of reasoning to that of the former article it may be shewn that the time of a complete small oscillation is

$$2\pi\sqrt{\frac{h^2+k^2}{gh}}$$

32. The formula for the time of a small oscillation of a compound pendulum may be written $2\pi\sqrt{\frac{l}{g}}$, where the value of l is $\frac{h^2+k^2}{h} = h + \frac{k^2}{h}$: this is called the length of the *equivalent simple pendulum.*

33. Experimental verification of the preceding formula:

(1) Rod moving in a vertical plane, about one end fixed, under gravity.

(2) Circular plate, oscillating about an axis through the rim perpendicular to the plane of the plate.

In these two cases, from the equation of energy, we may find the angular velocity in any position; and, also, determine the time of a small oscillation.

$$\text{For the rod, } T=2\pi\sqrt{\frac{4a}{3g}}$$

$$\text{For the plate, } T=2\pi\sqrt{\frac{3a}{2g}}$$

34. Since $\frac{1}{2}I\omega^2=Mgh\,(\cos\theta-\cos\alpha)$ we get, by differentiation,

$$I\frac{d\omega}{dt}=Mgh\sin\theta$$

and therefore the angular acceleration is equal to the moment of the external force of gravity divided by the moment of inertia.

The product $I\frac{d\omega}{dt}$ which represents the moment of inertia multiplied by the angular acceleration is sometimes termed the *Torque* or twisting effect of gravity on the body.

35. *Examples.*

(1) A rod of length 2 feet is fixed at one end, and falls, in a vertical plane, from a horizontal to a vertical position: find its angular velocity in the lowest position.

$4\sqrt{3}$ radians.

(2) Shew also that the linear velocity of the centroid is 6.92 feet per second.

(3) If the rod is hanging freely shew that the angular velocity necessary to raise it into a vertical position is $4\sqrt{6}$.

(4) A rectangular plate, sides 18″ and 22″, oscillates about an axis distant 9″.5 from its centroid: shew that the time of a small oscillation is 0″.65.

(5) A rod of length 2 feet makes small oscillations when suspended freely from one end: shew that the time is 0″.641.

36. Since the time of an oscillation of any heavy body about a fixed horizontal axis is equal to $\pi\sqrt{\frac{h^2+k^2}{gh}}$

it is evident that it must be a function of h, since k and g are constants. The time therefore will vary with the position of the axis with reference to the centroid, and will be a maximum when $h=0$ and a minimum when $\frac{dt}{dh}=0$ or when $h=k$. The variation of t in the case of a circular plate about a horizontal axis perpendicular to its plane is shewn in the following table:

Circular plate:

$$\text{radius} = 9'' = \tfrac{3}{4}'.$$

$$k^2 = \frac{a^2}{2} = \frac{9}{32}$$

h	$\frac{k^2}{h}$	$l = h + \frac{k^2}{h}$	t	
$\frac{3}{4}$	$\frac{3}{8}$	1.125	0.589	
$\frac{2}{3}$	$\frac{27}{64}$	1.088	0.579	
$\frac{7}{12}$	$\frac{27}{56}$	1.065	0.573	
$k = \frac{3\sqrt{2}}{8}$	$\frac{3\sqrt{2}}{8}$	1.061	0.570	Minimum value
$\frac{1}{2}$	$\frac{9}{16}$	1.062	0.571	
$\frac{5}{12}$	$\frac{27}{40}$	1.092	0.580	
$\frac{1}{3}$	$\frac{27}{32}$	1.177	0.600	
$\frac{1}{4}$	$\frac{9}{8}$	1.375	0.650	
$\frac{1}{6}$	$\frac{27}{16}$	1.854	0.756	
$\frac{1}{12}$	$\frac{27}{8}$	3.458	1.030	
0	∞	∞	∞	

The units are 1 foot and 1 second.

37. *Centre of suspension* and *centre of oscillation.*

For every body capable of rotation about a horizontal axis, under gravity, there exists two points which are reversible, when we consider the time of a small oscillation.

Illustrate by reference to a straight uniform rod, furnished with two knife edges, one at the end and the other at the centre of oscillation $\left(\frac{4a}{3}\right)$

Kater's pendulum for finding the value of gravity.

38. In general, when any body rotates about an axis under the action of external force, so long as this axis does not pass through the centre of gravity, the time of rotation from rest will be determined from the equation of energy $\frac{1}{2}I\omega^2 = W$
W being the work done.

$$\therefore \omega = \sqrt{\frac{2W}{I}}$$

$$\therefore \Delta\theta = \sqrt{\frac{2W}{I}}\,\Delta t$$

$$\therefore \Delta t = \sqrt{I}\,.\,\frac{\Delta\theta}{\sqrt{2W}}$$

$$\therefore t = \sqrt{I}\int\frac{d\theta}{\sqrt{2W}}$$

that is, the time to any particular position varies as the square root of the moment of inertia, whatever be the relation (supposed finite and real) between W and θ.

Illustrate the above relation by reference to the Torsion pendulum and the Balance wheel of a clock or watch.

39. In the case where the axis of rotation passes through the centroid, and the angular velocity changes uniformly, the formulas to be employed are similar to those for uniform linear acceleration:
those for uniform linear acceleration:

$$\omega_t = \omega^o + at \qquad (1)$$
$$\theta = \omega_o t + \tfrac{1}{2}at^2 \qquad (2)$$
$$\omega^2_t = \omega_o^2 + 2a\theta \qquad (3)$$

where ω = angular velocity at any time
ω_o = initial angular velocity
θ = angle described in any time
a = uniform angular acceleration

The relation (3) is obtained from (1) and (2), and (2) is obtained from (1) by integration. Relation (3) is really the equation of energy, for, multiplying both sides by $\frac{1}{2}I$, we get

$$\frac{1}{2}I\omega t^2 - \frac{1}{2}I\omega_o^2 = I\alpha\theta$$

and therefore the change of kinetic energy or the work done in time t is qual to $I\alpha\theta$.

Illustrative examples.

(1) A wheel weighing 81 lbs., whose radius of gyration is 8 inches, is acted on by a couple, whose moment is 5 lb. ft. units, for half a minute; find the angular velocity produced.

1271.2 revolutions per minute.

(2) Find the torque which, in one minute, will stop the rotation of a wheel of mass 160 lbs., radius of gyration 1 ft. 6 in., and angular velocity 10 turns per second. Find also the number of turns the wheel will take before stopping.

11.8 lbs. acting on an arm of 1 ft.
300 turns.

(3) A wheel, whose moment of inertia is 50, has a horizontal axle 4 inches in diameter, round which a cord is wrapped to which a 10 lb. weight is hung. How long will the weight take to descend 12 feet?

11.65 seconds.

(4) A cylindrical shaft 4 inches in diameter, weighing 84 lbs., turns without appreciable friction about a horizontal axis. A fine cord is wrapped around it by which a 20 lb. weight hangs. How long will the weight take to descend 12 feet?

1.25 seconds.

(5) Two weights, 3 lbs. and 5 lbs., hang over a. frictionless pulley in the form of a circular disc, whose weight is 12 ozs. Shew that the time taken by either weight to move through a distance of $\frac{64}{67}$ feet from rest is $\frac{1}{2}$ sec.

N.B. The motion is not dependent on radius but on weight of pulley.

40. *Impulsive Forces.*

An impulsive force is a very great force acting for a very short time, and may be measured by a change of momentum.

If, for example, a body be moving in a straight line with a velocity v, and this be changed suddenly to v', *in* *the same straight line*, then the value of the impulsive force which produced the change of velocity is $m\ (v'-v)$ where m is the mass of the body and v' is supposed to be greater than v.

Shew experimentally the difference between a finite and impulsive force.

41. *Action of an impulsive force* on a rod hanging freely from one end.

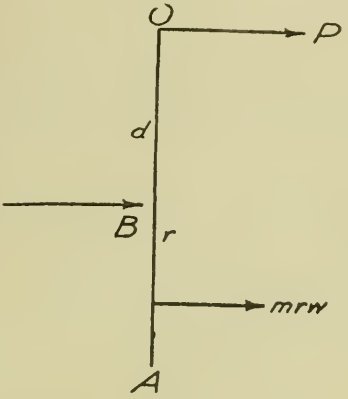


Fig. 25

Let the rod OA be struck a blow, B, at a point distant d from the axis of rotation, and in consequence of the action of this blow, let each elemental mass of the rod have a momentum $mr\omega$, where ω is the angular velocity produced suddenly by the blow.

Then evidently there will be some kind of impulsive pressure on O, which we may denote by P.

Then we must have

$$B = P + \Sigma(mr\omega) \qquad (1)$$

and if, in addition, we assume that the principle of moments applies to impulsive forces, we get

$$B \times d = \Sigma(mr^2\omega) = \omega\ .\ \Sigma(mr^2) \qquad (2)$$

From these two equations it will be seen that

$$\omega = \frac{3Bd}{4Ma^2}$$

$$\therefore P = B - Ma\omega$$

$$= B - \frac{3Bd}{4a}$$

$$\therefore P = B\left\{1 - \frac{3d}{4a}\right\}$$

and therefore P will be positive, negative, or zero, according as

$$d < > = \frac{4a}{3}.$$

42. Illustrate the preceding relation by means of experiments.

43. When $d = \frac{4a}{3}$, there is no impulsive pressure on the axis through O, and the point at which the blow is struck is called the *centre of percussion*.

44. Bodies, capable of motion about an axis, and symmetrical with respect to a line, have a centre of percussion on this line distant $\frac{k^2+h^2}{h}$ from the axis: where h is the distance from the centre of gravity to the axis and k is the radius of gyration.

45. Composition of angular velocities.
The phenomena of *precession* and *nutation*.
Elementary theory of the top and gyroscope.

APPENDIX

PROPERTIES OF THE CENTROID

✓ 1. *Statical.* Point in a body where the mass may be collected, or where the weight may be supposed to act.

✓ 2. *Dynamical.* Point about which rotation takes place most easily. Point of minimum effort or of minimum moments of inertia.

✓ 3. For all lines or planes drawn through the Centroid $\Sigma(mx)=0$.

✓ 4. From (3) we deduce the formulas for determining the position of the Centroid:

$$\bar{x}=\frac{\Sigma(mx)}{\Sigma(m)}$$

$$\bar{y}=\frac{\Sigma(my)}{\Sigma(m)}$$

5. Stability determined by the position of the Centroid.

6. Computation of work done in moving a *heavy* body from one position to another.

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